MATH 140, Section K2
November 18, 2009

Test #3

Name: Key

READ AND FOLLOW ALL DIRECTIONS. CIRCLE YOUR FINAL ANSWERS. SHOW ALL WORK TO RECEIVE FULL CREDIT. NO CALCULATORS.

1. (20 points) Let $f(x) = 4 - x$ and $g(x) = 1 + x^2$

   (a) Find $(f \circ g)(x)$.

   $= f(g(x)) = f(1+x^2) = 4 - (1+x^2) = \boxed{3 - x^2}$

   (b) Find $(g \circ f)(x)$.

   $= g(f(x)) = g(4-x) = 1 + (4-x)^2 = 1 + (16 - 8x + x^2)$
   $= \boxed{x^2 - 8x + 17}$

   (c) Find $(f \circ f)(4)$.

   $f \circ f(4) = f(f(4)) = f(4-4) = f(0) = 4 - 0 = \boxed{4}$

   (d) Find $(g \circ g)(-1)$.

   $g \circ g(-1) = g(g(-1)) = g(1 + (-1)^2) = g(2) = 1 + 2^2 = \boxed{5}$

2. (7 points) Determine whether or not each of the given functions is one-to-one. Explain your reasoning.

   (a) $\{(1, 2); (3, 5); (5, 8); (6, 10)\}$

   Yes. One-to-one because for each output there is exactly one input.

(b) $f(x) = 3x^2 + 2x + 5$

No. Not one-to-one; since the graph of f(x) is a parabola, which fails the vertical line test

3. (15 points) The function $f(x) = \frac{2-x}{3+x}$ is one-to-one.

(a) Find its inverse function $f^{-1}(x)$.

$$f(f^{-1}(x)) = x$$

$$\frac{2 - f^{-1}(x)}{3 + f^{-1}(x)} = x$$

$$2 - f^{-1}(x) = x(3 + f^{-1}(x))$$

$$2 - f^{-1}(x) = 3x + x f^{-1}(x)$$

$$-3x + f^{-1}(x) \qquad -3x + f^{-1}(x)$$

$$2 - 3x = x f^{-1}(x) + f^{-1}(x)$$

$$= (x+1) f^{-1}(x)$$

$$\boxed{f^{-1}(x) = \frac{2-3x}{x+1}}$$

Thrown out

(b) Check your answer by verifying that $(f^{-1} \circ f)(x) = x$

$$\cancel{f^{-1}(f(x)) = \frac{2 - \left(\frac{2-3x}{x+1}\right)}{3+}}$$

$$f^{-1}(f(x)) = \frac{2 - 3\left(\frac{2-x}{3+x}\right)}{\frac{2-x}{3+x} + 1} = \frac{2 - \frac{6-3x}{3+x}}{\frac{2-x}{3+x} + \frac{3+x}{3+x}} = \frac{\frac{2(3+x)}{3+x} - \frac{6-3x}{3+x}}{\frac{5}{3+x}}$$

$$= \frac{\frac{6 + 2x - 6 + 3x}{3+x}}{\frac{5}{3+x}} = \frac{5x}{5} = x$$

4. (20 points) Solve for x:

(a) $25^{2x} = 5^{x^2-14} \cdot 25$

$(5^2)^{2x} = 5^{x^2-14} \cdot 5^2$

$5^{4x} = 5^{x^2-12}$

$4x = x^2 - 12$

$0 = x^2 - 4x - 12$

$= (x-6)(x+2)$

$\boxed{x = 6 \text{ or } x = -2}$

(b) $\log_6(x+3) + \log_6(x+4) = 1$

$\log_6\left((x+3)\cdot(x+4)\right) = 1$

$6^{\log_6((x+3)\cdot(x+4))} = 6^1$

$(x+3)(x+4) = 6$

$x^2 + 7x + 12 = 6$

$x^2 + 7x + 6 = 0$

$(x+6)(x+1) = 0$

$\cancel{x = -6}$ or $x = -1$

Discard $x = -6$ since $x > -3$ and $x > -4$ (domain of log terms)

$\boxed{x = -1}$

5. (8 points) Find the exact value of each of the following expressions.

(a) $\log_2\left(\frac{1}{8}\right)$

$= \log_2\left(\frac{1}{2^3}\right) = \log_2\left(2^{-3}\right) = \boxed{-3}$

(b) $2^{\log_2 0.4} = \boxed{0.4}$

(c) $\ln(e^{\sqrt{2}})$ $= \log_e e^{\sqrt{2}} = \boxed{\sqrt{2}}$

(d) $\log_6 9 + \log_6 4$ $= \log_6(9 \cdot 4) = \log_6(36) = \log_6(6^2) = \boxed{2}$

6. (20 points) Write each expression as a single logarithm.

(a) $-2\log_3 \frac{1}{x} + \frac{1}{3}\log_3 x^3$

$= \log_3 \left(\frac{1}{x}\right)^{-2} + \log_3 (x^3)^{1/3}$

$= \log_3 x^{(-1)(-2)} + \log_3 x^{3 \cdot \frac{1}{3}}$

$= \log_3 x^2 + \log_3 x = \boxed{\log_3 x^3}$

(b) $\log(x^2 - 1) - 2\log(x+1)$

$= \log(x^2 - 1) - \log(x+1)^2$

$= \log \frac{(x^2-1)}{(x+1)^2}$ OR $= \log\left(\frac{x^2-1}{(x+1)^2}\right) = \log \frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(x+1)}$

$= \log \frac{x-1}{x+1}$

7. (10 points) For an exponential function $f(x) = a^x$, we require $a > 0$ and $a \neq 1$. Explain why this is so.

• $a \neq 1$ since $f(x) = 1^x = 1$ is a rather boring function.

• $a > 0$ since, for example, if $a = -2$,
$f(x) = (-2)^x$, and we would not have any ~~even~~ way to evaluate $f(\frac{1}{2})$, $f(\frac{1}{4})$, etc.
$a > 0$ to allow the domain to be all real numbers.

8. (5 points) EXTRA CREDIT. Prove that for $M, N > 0$, $a > 0$, $a \neq 1$, $\log_a(M \cdot N) = \log_a M + log_a N$. You may use any of the other rules for logs, except the one you are trying to prove. (Hint: Your proof should start with $\log_a(M \cdot N) = \ldots$ and follow a string of equalities to arrive at $\ldots = \log_a M + log_a N$.)

"Easy" Way

$$\log_a(M \cdot N) = \log_a\left(\frac{M}{N^{-1}}\right) = \log_a M - \log_a N^{-1}$$
$$= \log_a M - (-1)\log_a N = \log_a M + \log_a N$$

"Hard" Way

Let $A = \log_a M$, $B = \log_a N$. Then

$$\log_a(M \cdot N) = \log_a(a^A \cdot a^B) = \log_a(a^{A+B}) = A + B = \log_a M + \log_a N$$

$\left(\text{Note } M = a^A, N = a^B \text{ since } \begin{array}{l} a^A = a^{\log_a M} = M \\ a^B = a^{\log_a N} = N \end{array}\right)$