MATH 166 Section Z
February 7, 2013

Exam #1

Name: Key

READ AND FOLLOW ALL DIRECTIONS. CIRCLE YOUR FINAL ANSWERS. SHOW ALL WORK TO RECEIVE FULL CREDIT.

1. (25 points) Suppose that a particle travels along the curve defined by $x = \frac{1}{4}\cos(4t)$, $y = 5 + \frac{1}{4}\sin(4t)$, where $0 \le t \le \frac{\pi}{2}$.

   (a) How far has the object traveled?

$$\left(\frac{dx}{dt}\right)^2 = \left(4 \cdot -\frac{1}{4}\sin(4t)\right)^2 = (-\sin(4t))^2 = \sin^2(4t)$$

$$\left(\frac{dy}{dt}\right)^2 = \left(4 \cdot \frac{1}{4}\cos(4t)\right)^2 = (\cos(4t))^2 = \cos^2(4t)$$

$$L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{\sin^2(4t) + \cos^2(4t)}\, dt$$

$$= \int_0^{\pi/2} \sqrt{1}\, dt \quad \text{since } \cos^2\theta + \sin^2\theta = 1 \text{ for any } \theta$$

$$= \int_0^{\pi/2} 1\, dt = t\Big|_0^{\pi/2} = \frac{\pi}{2}$$

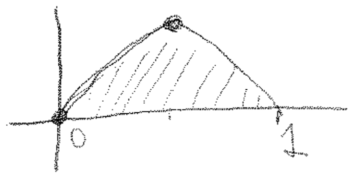
   (b) If the particle's path is rotated around the x-axis to create a surface, what integral would give the surface area? Set up, but *do not evaluate*.

$$A = \int_0^{\pi/2} y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/2} \left(5 + \frac{1}{4}\sin(4t)\right) dt$$

Exam #1

2. (25 points) (a) Sketch the region $\mathcal{R}$ bounded by the graph of $y = x - x^2$ and the x-axis. Find the area of the region.



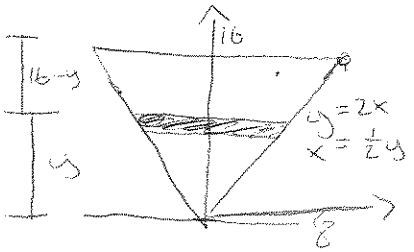
$$A = \int_0^1 (x - x^2)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1$$
$$= \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \boxed{\frac{1}{6}}$$

(b) A solid is generated by rotating $\mathcal{R}$ about the x-axis. Find the volume of the solid. (Give an exact answer.)

Use the disk method

$$V = \pi \int_0^1 (x - x^2)^2\,dx$$
$$= \pi \int_0^1 (x^2 - 2x^3 + x^4)\,dx$$
$$= \pi \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1$$
$$= \pi \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right] = \pi\left(\frac{10-15+6}{30}\right) = \boxed{\frac{\pi}{30}}$$

3. (25 points) A conical tank with vertex toward the ground has a height of 16 feet and a radius at the base of 8 feet. The tank is filled to the top with olive oil, which has a density of 53 lb/ft^3. How much work does it take to pump all of the oil to the rim of the tank? (HINT: Draw a picture! Round to the nearest whole number.)



$\delta = 53\ \text{lb}/\text{ft}^3$

limits of integration: 0 ft to 16 ft

$$\Delta W = \delta \cdot \Delta V \cdot \text{height lifted}$$

$$= \delta\left(\pi \cdot \left(\tfrac{1}{2}y\right)^2 \Delta y\right) \cdot (16-y)$$

$$= \frac{\pi\delta}{4} y^2 (16-y)\,\Delta y$$

$$W = \int_0^{16} \frac{\pi\delta}{4} y^2 (16-y)\,dy$$

$$= \frac{\pi\delta}{4}\int_0^{16} (16y^2 - y^3)\,dy$$

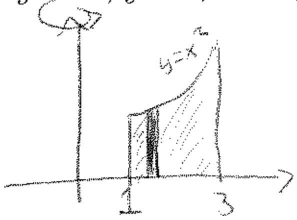
$$= \frac{\pi\delta}{4}\left[\frac{16y^3}{3} - \frac{y^4}{4}\right]_0^{16} = \frac{\pi\delta}{4}\left[\frac{16^4}{3} - \frac{16^4}{4}\right]$$

$$= \frac{\pi\delta}{4} \cdot 16^4\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{\pi\delta}{4} \cdot 16^4 \cdot \left(\frac{1}{12}\right)$$

$$= \frac{16^3\pi\delta}{3} \approx 227{,}334 \text{ ft-\#}$$

4. (25 points) Find the volume of the solid generated by revolving the region bounded by $y = x^2$, $y = 0$, $x = 1$, and $x = 3$ around the y-axis.



• use the shell method, slicing vertically
(washers would require horizontal slices and <u>two</u> different integrals)

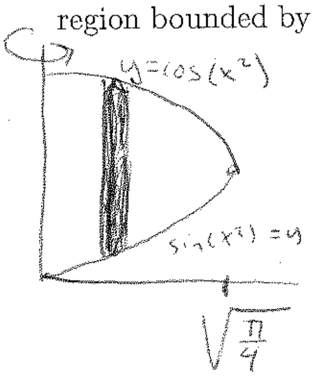
Avg radius: x
height: x^2
thickness: Δx
each slice

$$V = 2\pi \int_1^3 x(x^2)\,dx$$

$$= 2\pi \int_1^3 x^3\,dx = 2\pi\left[\frac{x^4}{4}\right]_1^3 = 2\pi\left[\frac{81-1}{4}\right]$$

$$= 2\pi\left[\frac{80}{4}\right] = \boxed{40\pi}$$

5. (5 points) EXTRA CREDIT. Find the volume of the solid obtained by rotating the region bounded by $y = \sin(x^2)$, $y = \cos(x^2)$, and $x = 0$ about the y-axis.



use the shell method with
Avg radius x, height $\cos(x^2) - \sin(x^2)$
and thickness Δx

$$V = \int_0^{\sqrt{\pi/4}} 2\pi x\left(\cos(x^2) - \sin(x^2)\right)dx$$

Let $u = x^2$
$du = 2x\,dx$
$0 \le u \le \frac{\pi}{4}$

$$= \int_0^{\pi/4} \pi\,(\cos u - \sin u)\,du$$

$$= \pi\left[\sin u + \cos u\right]_0^{\pi/4} = \pi\left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - 1\right] = \pi(\sqrt{2}-1)$$