Structure Theory of Commutative, Idempotent Groupoids of Bol-Moufang Type

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Motivation

To every finite algebra **A**, we can associate the decision problem CSP(**A**) (a *constraint satisfaction problem*).

Definition

An idempotent operation is a **weak near-unanimity (WNU)** operation if it satisfies

$$f(y, x, \dots, x) = f(x, y, x, \dots, x) = \cdots = f(x, x, \dots, y)$$

Theorem (Bulatov, Jeavons, Krokhin '05; Maroti & McKenzie '08)

Let **A** *be a finite idempotent algebra. If* **A** *has no weak near-unanimity term (WNU), then* CSP(**A**) *is NP-complete.*

Motivation

Algebraic Dichotomy Conjecture

If **A** has a WNU term, then CSP(**A**) is tractable.

• A binary term is a WNU iff it is commutative and idempotent.
• An algebra **A** with an associative binary WNU (semilattice term) has CSP(**A**) tractable.
• If the Algebraic Dichotomy Conjecture is true, any weakening of associativity (with C,I) should also suffice for tractability.

CI-Groupoids

Definition

Let $\mathbf{A} = \langle A, \cdot \rangle$ be a groupoid. We call $\mathbf{A}$ a *CI-groupoid* if $\cdot$ is both commutative and idempotent. Usually, we write xy for $x \cdot y$.

The Moufang Law $x(y(zy)) \approx ((xy)z)y$ is one weakening of associativity.

Definition

An identity $p \approx q$ is of *Bol-Moufang type* if (i) the only operation in p, q is $\cdot$, (ii) the same three variables appear on both sides, in the same order, (iii) one of the variables appears twice (iv) the remaining two variables appear only once.

Identities of Bol-Moufang Type (Phillips and Vojtěchovský)

A	$xxyz$
B	$xyxz$
C	$xyyz$
D	$xyzx$
E	$xyzy$
F	$xyzz$

1	$o(o(oo))$
2	$o((oo)o)$
3	$(oo)(oo)$
4	$(o(oo))o$
5	$((oo)o)o$

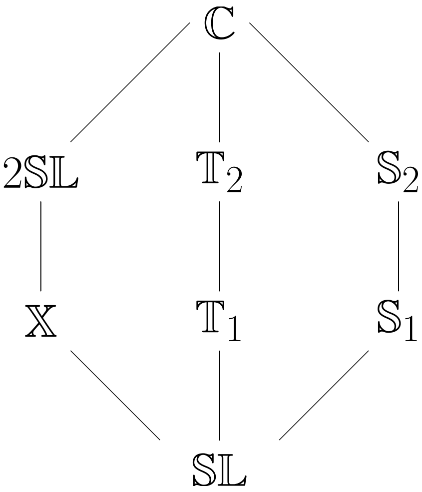
- Representable as Xij, the identity with:
  - variable order X
  - LHS bracketed by i, and RHS bracketed by j.
- $x(y(zy)) = ((xy)z)y$ is represented as $E15$.
- There are $6 * (4 + 3 + 2 + 1) = 60$ nontrivial such identities.

- A variety of ________________ is of *Bol-Moufang type* if it is axiomatized by one additional identity of Bol-Moufang type.
- Two identities of BM type are *equivalent* if they axiomatize the same variety of ________________ of B-M type.
- Phillips and Vojtěchovský showed there are 26 varieties of quasigroups, and 14 varieties of loops of B-M type.
- Q: How many varieties of CI-Groupoids of B-M type are there? What structure do they have?
    - Checking pairwise equivalence by hand would take too long!

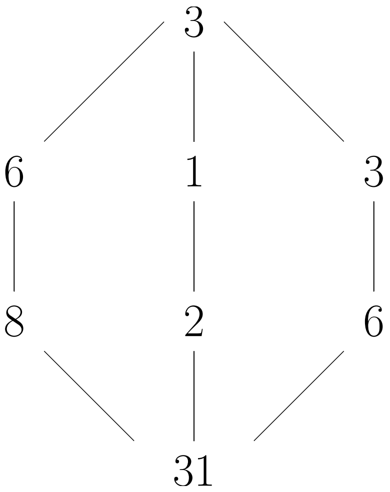
Prover9 / Mace4

- **Prover9** is an automated theorem prover for first-order and *equational logic*.
- **Mace4** searches for finite models and *counterexamples*.
- **Input:** a set of *assumptions* and one or more *goals*.
- **Output:** either *proofs* of the goals, or *counterexample(s)* where the assumptions are true but the goals are false.

The 8 Varieties of CI-Groupoids of Bol-Moufang Type



The 8 Varieties of CI-Groupoids of Bol-Moufang Type



SD(∧) Varieties

Definition

An algebra is *congruence meet-semidistributive* (SD(∧)) if its congruence lattice satisfies

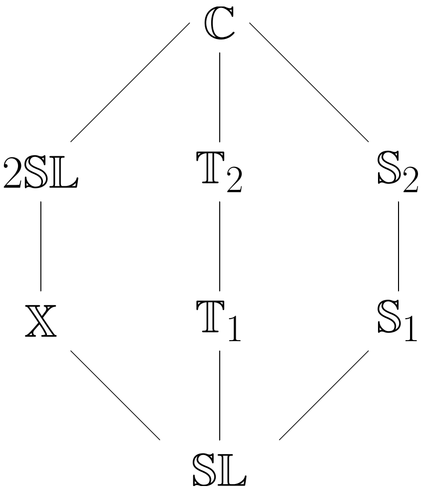
$$(x \wedge y \approx x \wedge z) \Rightarrow (x \wedge (y \vee z) \approx x \wedge y).$$

A class $\mathcal{K}$ is SD(∧) if every algebra in $\mathcal{K}$ is SD(∧).

The Universal Algebra Calculator can test if a finite algebra is SD(∧), or if it has WNU terms.

A finite, idempotent algebra which is SD(∧) is known to have tractable CSP.

The 8 Varieties of CI-Groupoids of Bol-Moufang Type



The Variety $\mathcal{S}_2$

Definition

$\mathcal{S}_2$ is the variety of CI-groupoids satisfying $x(y(xz)) \approx x((yx)z)$.

Theorem (KKVW '13)

A finite idempotent algebra with WNU terms $v(x, y, z)$ and $w(x, y, z, u)$ such that $v(y, x, x) \approx w(y, x, x, x)$ is $\mathrm{SD}(\wedge)$.

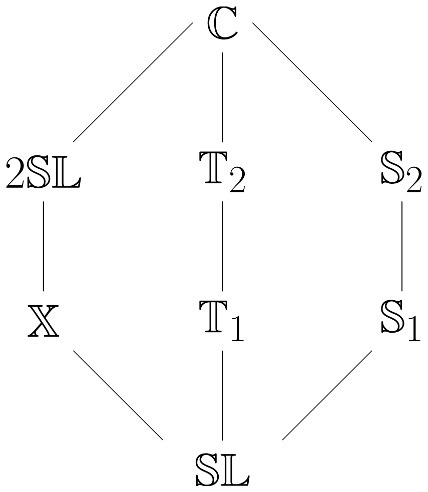
Theorem

$(\mathcal{S}_2)_{fin}$ *is* $\mathrm{SD}(\wedge)$.

Proof.

$\mathcal{S}_2$ has WNU terms $v(x, y, z) = (xy)(z(xy))$ and $w(x, y, z, u) = (xy)(zu)$ such that $v(y, x, x) \approx w(y, x, x, x)$. □

The 8 Varieties of CI-Groupoids of Bol-Moufang Type



Definition

Given

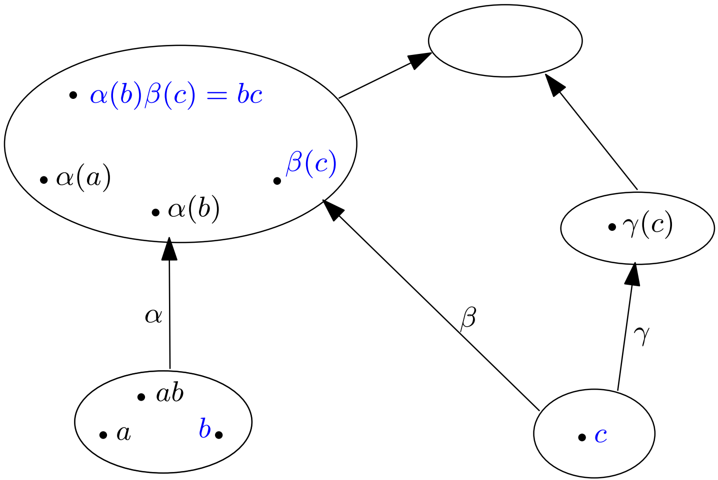
- $\mathbf{S} = \langle S, \vee \rangle$ a semilattice,
- $\{\mathbf{A}_s \mid s \in S\}$ a set of groupoids, and
- $\{\phi_{s,t} : \mathbf{A}_s \to \mathbf{A}_t \mid s \leq_\vee t\}$ a set of "nice" homomorphisms,

the **Płonka sum** over S of the groupoids $\{\mathbf{A}_s : s \in S\}$ is the groupoid $\mathbf{A}$ with universe $\biguplus_{s \in S} A_s$ and multiplication given by:

$$x_1 *^{\mathbf{A}} x_2 = \phi_{s_1,s}(x_1) *^{\mathbf{A}_s} \phi_{s_2,s}(x_2)$$

where $x_i \in \mathbf{A}_{s_i}$, $s = s_1 \vee s_2$.

The Płonka Sum of Groupoids



Płonka's Theorem

Theorem

Let $\mathcal{V}$ be the variety of groupoids defined by $\Sigma \cup \{x \vee y \approx x\}$ for some term $x \vee y$ and set Σ of regular identities. The following classes of algebras coincide:

(1) *The class* $\mathbf{Pl}(\mathcal{V})$*of Płonka sums of $\mathcal{V}$-algebras.*

(2) *The regularization, $\widetilde{\mathcal{V}}$, of $\mathcal{V}$.*

(3) *The variety of algebras of type ρ defined by the identities Σ and the following identities:*

$$x \vee x \approx x \tag{P1}$$
$$(x \vee y) \vee z \approx x \vee (y \vee z) \tag{P2}$$
$$x \vee (y \vee z) \approx x \vee (z \vee y) \tag{P3}$$
$$x \vee (y * z) \approx x \vee y \vee z \tag{P4}$$
$$(x * y) \vee z \approx (x \vee z) * (y \vee z) \tag{P5}$$

Squags and $\mathcal{T}_1$

Definition

$\mathcal{T}_1$ is the variety of CI-groupoids satisfying $x(x(yz)) \approx (x(xy))z$.

Definition

The variety $\mathcal{S}q$ of *Steiner quasigroups (squags)* is the variety of CI-groupoids satisfying $y(xy) \approx x$.

Theorem

$\mathcal{T}_1$ is the regularization of the variety of squags.

Proof.

Let $x \vee y \approx y(xy)$, $\Sigma = \{C, I, x(x(yz)) \approx (x(xy))z\}$. The squag identity ($x \vee y = x$) implies the $\mathcal{T}_1$ identity, so $\mathcal{S}q \subseteq \mathcal{T}_1$.
In $\mathcal{T}_1$, the term $x \vee y$ satisfies (P1)–(P5).
Thus, by Płonka's Theorem, $\mathcal{T}_1 = \widetilde{\mathcal{S}q} = \mathbf{Pl}(\mathcal{S}q)$. □

CID and CIE Groupoids

Definition

A groupoid is *distributive (D)* if it satisfies $x(yz) \approx (xy)(xz)$. It is *entropic (E)* if it satisfies $(xy)(zw) \approx (xz)(yw)$.

- Ježek, Kepka, and Němec: "the deepest non-associative theory within the framework of groupoids" is the theory of distributive groupoids.

Theorem

Every finite CID-groupoid (and hence CIE-groupoid) is a Płonka sum of quasigroups.

Further Work

- Paper: C. Bergman and D. Failing, *Commutative, idempotent groupoids and the constraint satisfaction problem.*
  - Submitted to A.U., preprint available at http://dfailing.public.iastate.edu
- More structure of $\mathcal{2SL}$, $\mathcal{X}$, $\mathcal{S}_2$ and $\mathcal{S}_1$, $\mathcal{T}_2$ and $\mathcal{T}_1$.
  - We stopped when the CSP was settled for each variety.
- Further generalizations?
  - Generalized Bol-Moufang CI-groupoids are either Bol-Moufang type or distributive.

Thanks!