Constraint Satisfaction Problems on Bol-Moufang CI-Groupoids

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Outline

What Are Constraint Satisfaction Problems?

Informally, a Constraint Satisfaction Problem (CSP) consists of a finite set of variables, ranging over some finite domain of values, and a set of constraints which restrict the values of the variables. The CSP asks whether there is an assignment of values to the variables such that all constraints are satisfied.

Examples

- Graph k-colorability.
- Solvability of systems of equations over a finite field.

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Definition

For any finite set A, and any set Γ of relations over A, **CSP**(Γ) is the combinatorial decision problem:

INSTANCE: A triple $\mathcal{R} = (V, A, \mathcal{C})$ where:

- V a finite set of *variables*
- $\mathcal{C} = \{(S_i, R_i) \mid i = 1, \ldots, n\}$ a set of *constraints*, with each S_i a tuple of variables, and each R_i an element of Γ which indicates the allowed simultaneous values for variables in S_i

QUESTION: Does $\mathcal{R}$ have a *solution*? That is, is there a map $f : V \to A$ such that for all i, $f(S_i) \in R_i$?

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Observation

CSP(Γ) is in NP.

Definition

If each instance of **CSP**(Γ) is answerable (yes/no) in polynomial time, we say that **CSP**(Γ) is **tractable**, or that Γ is a tractable set of relations. Γ is NP-complete if there is some finite $\Delta \subseteq \Gamma$ for which $CSP(\Delta)$ is NP-complete.

Examples

- For k-colorability, if $k = 2$ we can use Breadth First Search to produce a coloring (or show none exists) in polynomial time. For $k \geq 3$, the problem is known to be NP-complete.
- For systems of linear equations over a finite field, Gaussian elimination will find a solution, if it exists, in polynomial time.

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CSP Dichotomy Conjecture (Feder and Vardi '98)

Every **CSP**(Γ) is either tractable, or it is NP-complete.

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Characterize *all* tractable sets of relations.

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Characterize *all* tractable sets of relations.

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We say that an m-ary operation $f\colon A^m \to A$ **preserves** an n-ary relation R over A (or that R is **invariant** under f) if

$$\bar{a}_1, \ldots, \bar{a}_m \in R \Rightarrow f(\bar{a}_1, \ldots, \bar{a}_m) \in R$$

For Γ a set of relations over A and $\mathcal{F}$ a set of operations on A:

$\mathrm{Pol}(\Gamma) := \{f \mid f \text{ preserves every } R \in \Gamma\}$,
the **clone of polymorphisms** of Γ.

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Definition

An **algebra** is a pair $\mathbf{A} = \langle A, \mathcal{F} \rangle$, where A is a nonempty set, and $\mathcal{F}$ is a set of operations on A.

Observation

To every set of relations Γ over a finite set A, we can associate the algebra $\mathbf{A} = \langle A, \mathrm{Pol}(\Gamma) \rangle$. Likewise, to every finite algebra $\mathbf{A} = \langle A, \mathcal{F} \rangle$, we can associate the set of relations $\mathrm{Inv}(\mathcal{F})$.

Definition

We say an algebra $\mathbf{A} = \langle A, \mathcal{F} \rangle$ is **tractable** if $\mathrm{Inv}(\mathcal{F})$ is a tractable set of relations. Similarly, $\mathbf{A}$ may be **NP-complete**. We can consider only idempotent algebras. $(\forall f \in \mathcal{F},\ f(x, x, \ldots, x) \approx x)$

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Definition

An idempotent operation on a set A is a **weak near-unanimity (WNU)** operation if it satisfies

$$f(y, x, \dots, x) = f(x, y, x, \dots, x) = \cdots = f(x, x, \dots, y)$$

Theorem (Bulatov, Jeavons, Krokhin '05; Maroti & McKenzie '08)

Let $\mathbf{A} = \langle A, \mathcal{F} \rangle$ *be a finite algebra. If* $\mathrm{Clo}(\mathcal{F})$ *contains no weak near-unanimity operation, then* $\mathbf{A}$ *is NP-complete.*

Algebraic Dichotomy Conjecture

If $\mathrm{Clo}(\mathcal{F})$ contains a WNU, then $\mathbf{A}$ is tractable.

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Definition

A ternary operation q is **Maltsev** if it satisfies

$$q(x, y, y) = q(y, y, x) = x$$

Example

For $\mathbf{G} = \langle G, \cdot, ^{-1}, e \rangle$ a group, $q(x, y, z) = x \cdot y^{-1} \cdot z$.

Example

For $\mathbf{Q} = \langle Q, \cdot, /, \backslash \rangle$ a quasigroup, $q(x, y, z) = (x/(y \backslash y)) \cdot (y \backslash z)$.

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Theorem (IMMVW '10)

If $\text{Clo}(\mathcal{F})$ *contains a Maltsev term, then* $\mathbf{A} = \langle A, \mathcal{F} \rangle$ *is tractable.*

Fact

If $\text{Clo}(\mathcal{F})$ contains a Maltsev term, then it also contains a WNU.

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Definition

Maroti-Janko terms are ternary r, s, t with s a WNU satisfying:

$$r(x,x,y) = r(x,y,x) = t(y,x,x) = t(x,y,x) = s(x,x,y)$$

$$r(y,x,x) = t(y,y,x)$$

Theorem (Barto)

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If $\mathrm{Clo}(\mathcal{F})$ *contains Maroti-Janko terms, then* $\mathbf{A} = \langle A, \mathcal{F} \rangle$ *is tractable.*

Definition

A **semilattice** operation is a binary operation which is associative, commutative, and idempotent.

Theorem (Jeavons, Cohen, Gyssens '97)

If Clo($\mathcal{F}$) *contains a semilattice operation, then* $\mathbf{A} = \langle A, \mathcal{F} \rangle$ *is tractable.*

Fact

Every semilattice operation $\cdot$ gives rise to a WNU term

$$f(x, y, z) = x \cdot (y \cdot z)$$

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Approaching The Dichotomy Conjecture

- A binary WNU $\cdot$ satisfies:
  1. $x \cdot x = x$ (idempotence)
  2. $x \cdot y = y \cdot x$ (commutativity)
- Neither alone is sufficient for tractability.
- Chose to study something more general than semilattice operation, but more structured than binary WNU.

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Definition

Let $\mathbf{A} = \langle A, \cdot \rangle$ be a groupoid. We call $\mathbf{A}$ a *CI-groupoid* if $\cdot$ is both commutative and idempotent. Usually, we write xy for $x \cdot y$.

Definition

An identity $p \approx q$ is of Bol-Moufang type if (i) the only operation in p, q is $\cdot$, (ii) the same three variables appear on both sides, in the same order, (iii) one of the variables appears twice (iv) the remaining two variables appear only once.

Example

The Moufang Law $x(y(zy)) = ((xy)z)y$ is an identity of Bol-Moufang type.

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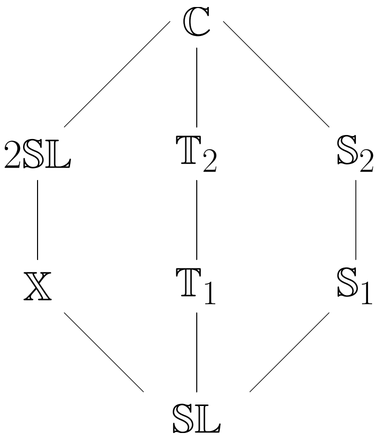
Identities of Bol-Moufang Type (Philips and Vojtěchovský)

A	$xxyz$
B	$xyxz$
C	$xyyz$
D	$xyzx$
E	$xyzy$
F	$xyzz$

1	$o(o(oo))$
2	$o((oo)o)$
3	$(oo)(oo)$
4	$(o(oo))o$
5	$((oo)o)o$

Each such identity is representable as X_{ij}, where:
$X \in \{A, ..., F\}$, $1 \leq i < j \leq 5$, the identity with variables ordered by X, whose LHS is bracketed according to i, and whose RHS is bracketed according to j.
Two identities are **equivalent** (relative to a variety) if they define the same subvariety.

The 8 Varieties of CI-Groupoids of Bol-Moufang Type



The 8 Varieties of CI-Groupoids of Bol-Moufang Type

$\mathbb{C}$	All CI-groupoids
$2\mathbb{SL}$	$x(xy) = xy$
$\mathbb{X}$	$A24$: $x((xy)z) = (x(xy))z$
$\mathbb{SL}$	Semilattices
$\mathbb{T}_2$	$C15$: $x(y(yz)) = ((xy)y)z$
$\mathbb{T}_1$	$A14$: $x(x(yz)) = (x(xy))z$
$\mathbb{S}_2$	$B12$: $x(y(xz)) = x((yx)z)$
$\mathbb{S}_1$	$B13$: $x(y(xz)) = (xy)(xz)$

The 8 Varieties of CI-Groupoids of Bol-Moufang Type

$\mathbb{C}$	Dichotomy Conjecture?
$2\mathbb{SL}$	Maroti-Janko Terms
$\mathbb{X}$	Maroti-Janko Terms
$\mathbb{SL}$	Jeavons, Cohen, Gyssens
$\mathbb{T}_2$	$C15$: $x(y(yz)) = ((xy)y)z$
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$\mathbb{S}_2$	Maroti-Janko Terms
$\mathbb{S}_1$	Maroti-Janko Terms

Definition

The class of CI-groupoids defined by the additional identity $x(yx) = y$ is known as the variety of Steiner quasigroups (*squags*).

Theorem

$\mathbb{T}_1$ contains the variety of squags.

Proof.

For squags, $x(x(yz)) = yz = (x(xy))z$, so $A14$ holds. □

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Definition

Let $\mathbf{S} = \langle S, \vee \rangle$ be a semilattice, considered as a category with morphisms $s \to t \Leftrightarrow s \leq t$ in S, V a variety of groupoids considered as a category, and $F\colon S \to V$ a functor. Then the **Płonka sum** over S of the groupoids $\{\mathbf{A}_s = F(s) : s \in S\}$ is the groupoid $\mathbf{A}$ with universe $\dot\bigcup_{s \in S} A_s$ and multiplication given by:

$$x_1 \cdot^{\mathbf{A}} x_2 = F_{s_1 s}(x_1) \cdot^{\mathbf{A}_s} F_{s_2 s}(x_2)$$

where $x_i \in \mathbf{A}_{s_i}$, $s = s_1 \vee s_2$, and $F_{s_i s} = F(s_i \to s)$

Płonka's Theorem

Let **V** be a variety of groupoids defined by identities $\Sigma \cup \{x * y = x\}$ for some set Σ of regular identities, and $x * y$ a binary term. The following classes of algebras coincide:

1. The class Pł(**V**) of Płonka sums of groupoids from **V**.
2. The variety of groupoids defined by Σ and the identities:

$$x * x = x$$
$$(x * y) * z = x * (y * z)$$
$$x * y * z = x * z * y$$
$$(xy) * z = (x * z)(y * z)$$
$$x * (yz) = x * y * z$$

Theorem

$\mathbb{T}_1$ *is the class of Płonka sums of squags.*

Proof.

- Let $\Sigma = \{xx = x, xy = yx, x(x(yz)) = (x(xy))z\}$, and $x * y := y(xy)$.
- For squags, $x * y = x$. Since $\mathbb{T}_1$ contains the variety of squags, it is enough to show that Σ entails each of the identities in the theorem.
- Ask Prover9 to do it for you. Verify by hand over several days. Celebrate.

□

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Proof.

- Let $\Sigma = \{xx = x, xy = yx, x(x(yz)) = (x(xy))z\}$, and $x * y := y(xy)$.
- For squags, $x * y = x$. Since $\mathbb{T}_1$ contains the variety of squags, it is enough to show that Σ entails each of the identities in the theorem.
- Ask Prover9 to do it for you. Verify by hand over several days. Celebrate.

□

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Theorem

Every finite algebra $\mathbf{A} \in \mathbb{T}_1$ *is tractable.*

Next Step - The Variety $\mathbb{T}_2$

Examples of Subdirectly Irreducible Algebras in $\mathbb{T}_2 \setminus \mathbb{T}_1$

	0	1	2	3	4	5	6
0	0	2	1	3	5	4	6
1	2	1	0	3	5	4	6
2	1	0	2	4	3	5	6
3	3	3	4	3	5	4	6
4	5	5	3	5	4	3	6
5	4	4	5	4	3	5	6
6	6	6	6	6	6	6	6

	0	1	2	3	4	5
0	0	2	1	3	5	4
1	2	1	0	3	5	4
2	1	0	2	4	3	5
3	3	3	4	3	5	4
4	5	5	3	5	4	3
5	4	4	5	4	3	5

Further Research

- Investigate *entropic* CI-groupoids: $(xy)(zw) = (xz)(yw)$.
- Investigate *distributive* CI-groupoids: $x(yz) = (xy)(xz)$.
- Investigate some of the other varieties of Bol-Moufang CI-Groupoids. Exactly *which* subvariety of $2\mathbb{SL}$ is defined by $\mathbb{X}$? Can we develop a structure theory for $\mathbb{X}$, $\mathbb{S}_1$ or $\mathbb{S}_2$?